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Cramer–Rao information plane of orthogonal hypergeometric polynomials

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Abstract

The classical hypergeometric polynomials $\{p_n(x)\}_{n=0}^{\infty}$, which are orthogonal with respect to a weight function $\omega(x)$ defined on a real interval, are analyzed in the Cramer–Rao information plane, that is the plane defined by both Fisher information and variance of the probability density $\rho_n(x) = p_n(x)^2\omega(x)$. The Rakhmanov density $\rho_n(x)$ of these polynomials, which describes the probability density of the quantum states for various physical prototypes in an exact manner and for numerous physical systems to a very good approximation, is discussed in detail.

MSC: 33C45; 42C05; 94A17; 62B10

Keywords: Information theory; Special functions; Classical orthogonal polynomials; Fisher information; Variance; Cramer–Rao information plane; Cramer–Rao inequalities

1. Introduction

The classical continuous hypergeometric polynomials $\{p_n(x); n = 0, 1, \ldots\}$ orthogonal with respect to the weight function $\omega(x)$ on a real interval (a, b) are standard objects of analysis which have been shown its great applicability not only in mathematics but also in numerous scientific and social sciences

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[1,5,14,19]. In particular, the three main sequences of the classical orthogonal polynomials (Hermite, Laguerre and Jacobi) control the wavefunctions of the quantum-mechanical states of numerous physical systems with a Schrödinger's equation analytically solvable or quasi-exactly solvable [3,14,20].

The distribution of the classical orthogonal polynomials $\{p_n(x)\}$ on their orthogonality interval (a, b) and the spreading of the associated probability density $\rho_n(x) = p_n^2(x)\omega(x)$ defined on (a, b), can be most appropriately analyzed by means of information-theoretic measures [11,17]. The density $\rho_n(x)$ is relevant from both mathematical and physical points of view. Indeed, it governs the behavior of the ratio p_{n+1}/p_n when $n \to \infty$, as Rakhmanov showed in his pioneering work [15]. Also, it describes the probability density of the quantum states of various physical prototypes, such as e.g., the harmonic oscillator, the hydrogenic atom and the rigid rotator [13], as well as it is a very good approximation for the quantum-mechanical density of the physical states of numerous physical systems [3,14,20].

The study of the spreading of the classical orthogonal polynomials by means of information-theoretic measures, was initiated in the 1990s with the computation of their Shannon information entropy [2,8,9]. The results found for this quantity up to 2001 are summarized in Ref. [7], where one can also see in detail the intimate connection of the classical orthogonal polynomials as well as their associated Rakhmanov densities $\rho_n(x)$ with some quantum problems. Just recently [16] the Fisher information has been analyzed for these systems. Contrary to the Shannon entropy case, which cannot be analytically calculated save for the Chebyshev polynomials of first and second class, the Fisher information has been calculated in a closed form for all classical orthogonal polynomials on a real interval (Hermite, Laguerre, Jacobi). This has been possible because both information measures have a qualitatively different character. The Shannon entropy is a logarithmic functional of the polynomials, so that it is a global measure of their spreading on the orthogonality interval. The Fisher information is a gradient functional of the polynomials, so that it is a local measure of the concentration of the polynomials.

In this work we extend and improve this study by use of two new information-theoretic tools, the Fisher-variance information product and the Cramer–Rao information plane (also called Fisher–Heisenberg plane for reasons that we explain later on). This plane is defined by two quantities, the Fisher information and the variance of the polynomials, as discussed in Section 2. Then, in Section 3 we study the Fisher information and the variance of the three main families of the classical orthogonal polynomials on a real interval. In Section 4, the Fisher information, the variance, the Cramer–Rao information plane and the information product are discussed in detail for all the three families of polynomials. Finally, some concluded remarks and various open problems are pointed out in Section 5.

2. The Cramer–Rao information plane

Let us consider a random variable X whose probability density function is denoted as $\rho(x)$, assumed to be unity normalized. Its Fisher information [11,12] is the expectation value of the squared logarithmic derivative of $\rho(x)$, i.e.,

$$I_X = \int_\Delta \rho(x)\left[\frac{\mathrm{d}}{\mathrm{d}x} \log \rho(x)\right]^2 \mathrm{d}x, \tag{1}$$

which may be written as

$$I_X = \int_\Delta \left[\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\right]^2 \frac{\mathrm{d}x}{\rho(x)} = 4\int_\Delta \left[\frac{\mathrm{d}}{\mathrm{d}x}\rho^{1/2}(x)\right]^2 \mathrm{d}x.$$

And the variance of X is given by

$$V_X = \int_\Delta (x - \langle x\rangle)^2 \rho(x)\,\mathrm{d}x = \langle x^2\rangle - \langle x\rangle^2, \tag{2}$$

where $\langle x^m\rangle$ denotes the expectation value of x^m.

These two quantities quantitatively measure the spreading of the variable X in a very different manner, because their analytical structures strongly differ one to another. The Fisher information is essentially a functional of the derivative of the density $\rho(x)$ so that it is very sensitive to local rearrangements of the variable. For the contrary, the variance is a global measure in a sense stronger than the Shannon entropy of $\rho(x)$. This is because the variance gives a larger weight to the tails of the distribution than the logarithmic Shannon functional. Needless to say that the strong dependence on the tails of the variance is not relevant when they fall off exponentially, as for a Gaussian distribution.

These information-theoretic measures have, among other characteristics, a number of properties [12] which deserve to be resembled here.

(i) Scaling property. The Fisher information and the variance transform as

$$I_{cX} = |c|^{-2} I_X,$$
$$V_{cX} = |c|^2 V_X,$$

when the variable is scaled by a scalar factor $c \in \mathbb{C}^*$.

(ii) Cramer–Rao inequality [6]. The two measures verify the inequality

$$I_X V_X \geqslant a, \tag{3}$$

where $a = 1$ when the support interval Δ of the random variable is $(-\infty, +\infty)$ or $[0, +\infty)$, and $a = 0$ for $\Delta = [-1, +1]$. Equality only occurs for the Gaussian, exponentially decreasing and uniform density, respectively.

(iii) Stam uncertainty property [10,18]. Denoting by $\tilde{X}$ the Fourier transform of the variable X, the Fisher information and the variance satisfy the two following uncertainty relationships,

$$I_X \leqslant 4V_{\tilde{X}},$$
$$I_{\tilde{X}} \leqslant 4V_X.$$

The last two properties lead to the celebrated Heisenberg uncertainty principle, i.e., $V_X V_{\tilde{X}} \geqslant 1/4$. Moreover, the scaling and Cramer–Rao relations show that both Fisher information and variance measures are closely connected, so that the characterization of the density $\rho(x)$ should be improved when analyzing their location in the Fisher information-variance plane (also called, for obvious reasons, Cramer–Rao plane or even Fisher–Heisenberg plane).

It is worthy pointing out two observations, which will play a role later on in this work. First, the densities associated to the random variable X and the scaled version aX, respectively, belong to the same

$I_X V_X$=constant curve of the plane. Second, the Gaussian limit $I_X V_X = 1$ defines the line above which lie down all the densities with values of Fisher's information and variance such that $I_X \geqslant 0$, $V_X \geqslant 0$ and $I_X V_X \geqslant 1$.

Finally, the Cramer–Rao inequality motivates us to introduce a new measure of information, the Fisher-variance information product

$$\mathscr{P}_X = I_X V_X - a, \tag{4}$$

where a has been defined above. This measure has the advantages of nonnegativeness (as Fisher information and variance) and scale invariance (as the Kullback–Leibler or relative entropy).

3. Fisher's information and variance of classical orthogonal polynomials

Let $\{p_n(x)\}$ denote a sequence of real polynomials orthogonal with respect to the weight function $\omega(x)$ on the interval $(a, b) \subseteq \mathbb{R}$, i.e.,

$$\int_a^b p_n(x) p_m(x) \omega(x) = \delta_{n,m};\ m, n \in \mathbb{N}$$

with deg $p_n(x) = n$. Let us also assume that the weight function $\omega(x)$ is nonnegative, i.e., $\omega(x) \geqslant 0$, $\forall x \in (a, b)$. Then the function

$$\rho_n(x) = p_n^2(x)\omega(x) \tag{5}$$

are normalized density functions for the continuous random variable X.

These polynomials are usually called hypergeometric-type polynomials, and can be reduced by means of linear changes of the variable to one of the three classical families, i.e., Hermite, Laguerre and Jacobi. They can be characterized in various forms [14]. Let us mention here (a) the differential equation

$$\sigma(x) p_n''(x) + \tau(x) p_n'(x) + \lambda_n p_n(x) = 0, \tag{6}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of second and first degree at most, respectively, and the scalar $\lambda_n = n\tau' + \frac{1}{2}n(n-1)\sigma''$, and (b) the three-term recurrence relation

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x), \tag{7}$$

where a_n and b_n are scalars. The coefficients of these two relationships for the three main families of classical orthogonal polynomials are given in Table 1.

The information measures of the polynomials $p_n(x)$ are defined as the information measures associated to the Rakmanov density function $\rho_n(x)$ given by Eq. (5), which provide a quantitative measure of the spreading/concentration of the polynomials on the real interval (a, b). Then, the Fisher information $I(p_n)$ of the classical orthogonal polynomials $p_n(x)$ is, according to Eq. (1), the integral

$$\begin{aligned} I(p_n) &= \int_a^b \rho_n(x) \left[\frac{\mathrm{d}}{\mathrm{d}x} \log \rho_n(x)\right]^2 \mathrm{d}x \\ &= \int_a^b p_n^2(x)\omega(x) \left\{\frac{\mathrm{d}}{\mathrm{d}x} \log [p_n^2(x)\omega(x)]\right\}^2 \mathrm{d}x. \end{aligned} \tag{8}$$

Table 1
Main data of the classical orthonormal polynomials

$p_n(x)$	$H_n(x)$	$L_n^\alpha(x);\ \alpha>-1$	$P_n^{(\alpha,\beta)}(x);\ \alpha>-1,\ \beta>-1$
(a,b)	$(-\infty,+\infty)$	$[0,+\infty)$	$[-1,+1]$
$\omega(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$\alpha+1-x$	$\beta-\alpha-(\alpha+\beta+2)x$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
a_n	$\sqrt{\frac{n+1}{2}}$	$-\sqrt{(n+1)(n+\alpha+1)}$	$\frac{2}{2n+\alpha+\beta+2}\sqrt{\frac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}}$
b_n	0	$2n+\alpha+1$	$\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$

This quantity has been recently calculated [16] in analytical form by use of the differential equation given by Eq. (6) and Table 1. The values of the Fisher information obtained are

$$I(H_n)=4n+2 \tag{9}$$

for the Hermite polynomials $H_n(x)$,

$$I(L_n^\alpha)=\begin{cases} 4n+1 & \text{for } \alpha=0,\\ \dfrac{(2n+1)\alpha+1}{\alpha^2-1} & \text{for } \alpha>1,\\ \infty & \text{for } \alpha\in(-1,1),\ \alpha\neq 0 \end{cases} \tag{10}$$

for the Laguerre polynomials $L_n^\alpha(x)$, $\alpha>-1$ and $n=0,1,2,\ldots$, and

$$I(P_n^{(\alpha,\beta)})=\begin{cases} \dfrac{2n+\alpha+\beta+1}{4(n+\alpha+\beta+1)}\left[n(n+\alpha+\beta-1)\left(\dfrac{n+\alpha}{\beta+1}+2+\dfrac{n+\beta}{\alpha+1}\right)\right. & \\ \quad\left.+(n+1)(n+\alpha+\beta)\left(\dfrac{n+\alpha}{\beta-1}+2+\dfrac{n+\beta}{\alpha-1}\right)\right] & \text{for } \alpha,\beta>1,\\ \dfrac{2n+\alpha+1}{4}\left[\dfrac{n^2}{\alpha+1}+n+(4n+1)(n+\alpha+1)+\dfrac{(n+1)^2}{\alpha-1}\right] & \text{for } \alpha>1,\ \beta=0,\\ 2n(n+1)(2n+1) & \text{for } \alpha,\beta=0,\\ \infty & \text{otherwise} \end{cases} \tag{11}$$

for the Jacobi polynomials $P_n^{(a,b)}(x)$; $\alpha,\beta>-1$ and $n=0,1,\ldots$. One should keep in mind that $I(P_n^{(\alpha,\beta)})=I(P_n^{(\beta,\alpha)})$ due to the (α,β) symmetry of Jacobi polynomials.

Since the Legendre polynomials $P_n(x)$ and the Gegenbauer polynomials $C_n^\lambda(x)$ are related to the Jacobi polynomials by means of the relations

$$P_n(x)=P_n^{(0,0)}(x)\ ,\ \ C_n^\lambda=P_n^{(\lambda-1/2,\lambda-1/2)}(x),$$

respectively, it is straightforward to evaluate from Eqs. (11) the Fisher information of these two Jacobi subfamilies. We have that

$$I(P_n)=2n(n+1)(2n+1) \tag{12}$$

for the Legendre polynomials, and

$$I(C_n^{\lambda}) = \frac{2n+2\lambda}{2}\left[\frac{n(n+2\lambda)}{\lambda+\frac{1}{2}} + \frac{(n+1)(n+2\lambda-1)}{\lambda-\frac{3}{2}}\right], \quad \alpha > 1, \tag{13}$$

for the Gegenbauer polynomials with parameter λ.

Let us highlight that in all the cases for which the Fisher information of Laguerre polynomials is finite, it is a linear increasing function of the degree n; so, it has the same qualitative asymptotic behavior as the Hermite polynomials. In the Jacobi case, the corresponding Fisher information grows as n^3 for large n.

The variance of the classical orthogonal polynomials is, according to Eq. (2), the integral

$$V(p_n) = \int_a^b (x - \langle x\rangle_n)^2 \rho_n(x)\,\mathrm{d}x = \langle x^2\rangle_n - \langle x\rangle_n^2, \tag{14}$$

where

$$\langle x\rangle_n = \int_a^b x p_n^2(x)\omega(x)\,\mathrm{d}x = b_n$$

$$\langle x^2\rangle_n = \int_a^b x^2 p_n^2(x)\omega(x)\,\mathrm{d}x = a_n^2 + b_n^2 + a_{n-1}^2$$

so that,

$$V(p_n) = a_n^2 + a_{n-1}^2$$

in terms of the coefficients of the recurrence relation (7). By use of the coefficients given in Table 1, we obtain for the variance of the classical orthogonal polynomials the values

$$V(H_n) = n + \tfrac{1}{2} \tag{15}$$

for the Hermite polynomials $H_n(x)$,

$$V(L_n^{\alpha}) = 2n^2 + 2(\alpha+1)n + \alpha + 1 \tag{16}$$

for the Laguerre polynomials $L_n^{\alpha}(x)$, and

$$V(P_n^{(\alpha,\beta)}) = \frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} \tag{17}$$

for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. We notice that the asymptotic (i.e., large n) behavior of the variance of the classical orthogonal polynomials behaves as n and $2n^2$ in the Hermite and Laguerre cases, respectively, and tends to $1/2$ in the Jacobi case.

4. The Cramer–Rao information plane of classical orthogonal polynomials

In this Section, we compute the Fisher information $I(p_n)$ and the variance $V(p_n)$ of the Hermite, Laguerre and Jacobi polynomials, and we describe the region spanned by these polynomials in the

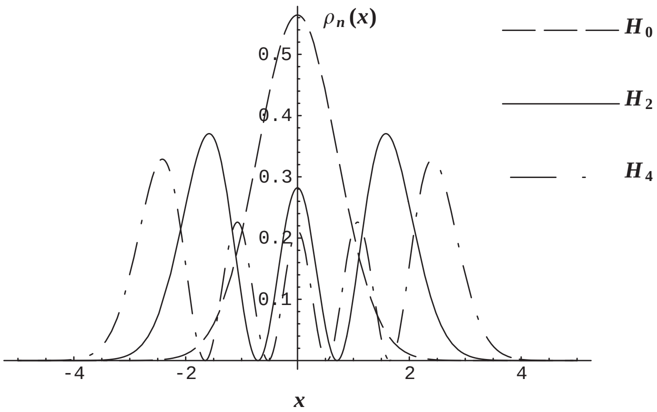

Fig. 1. Rakhmanov density of the Hermite polynomials $H_n(x)$ with $n = 0, 2$ and 4.

Cramer–Rao information plane, i.e., the set of all reachable values (I, V) for a specific class of polynomials. In addition keeping in mind Eqs. (4) and (5), we define and analyze a new information measure to study the spreading of the Rakhmanov densities of the classical orthogonal polynomials, and consequently of the polynomials themselves: the information product $\mathscr{P}(p_n) = I(p_n)V(p_n) - a$.

4.1. *Hermite polynomials $H_n(x)$*

According to Eqs. (9) and (15) we have that both the Fisher information $I(H_n)$ and the variance $V(H_n)$ grow linearly with the degree *n* of the polynomials. Moreover, it is observed that $I(H_n) = 4V(H_n)$, which can be verified in a straightforward and analytical way. In addition, the region of Hermite (I, V) points in the Cramer–Rao information plane appear as a discrete line of bullets as one can see in the information plane of the classical orthogonal polynomials given later on (see Fig. 22). Finally let us write down the information product $\mathscr{P}(H_n) = (2n + 1)^2 - 1$. Notice that either of these information measures can be reciprocally used to determine the degree of the corresponding polynomial. For example, the degree *n* of a polynomial with a given Fisher information is equal to $(I - 2)/4$.

The behavior of these information-theoretic measures can be understood by looking at Fig. 1, showing the Rakhmanov density $\rho_n(x) = H_n^2(x) \exp(-x^2)$ (see Eq. (5)) of the Hermite polynomials with degrees $n = 0, 2$ and 4. It is apparent that when *n* is increasing, the oscillations of the density regularly and symmetrically enhance (so, the Fisher informations increases linearly) and its spread around the origin gets larger (so, its variance also increases linearly). The information product naturally reflects the same phenomena, but considerably enhanced (Fig. 2).

4.2. *Laguerre polynomials $L_n^\alpha(x)$, $\alpha > -1$*

The Fisher information of the Laguerre polynomials $I(L_n^\alpha)$ with a fixed parameter α is finite only when $\alpha=0$ and $\alpha > 1$, as seen in the previous section. In both cases the Fisher information linearly depends on the degree of the polynomials, while the variance (which is finite for all values of $\alpha > -1$) shows a quadratic behavior with *n* as it is illustrated in Figs. 3 and 4, respectively. We observe that the Fisher information of the Laguerre polynomials, see Fig. 3, has practically the same variation with *n* than in the Hermite case,

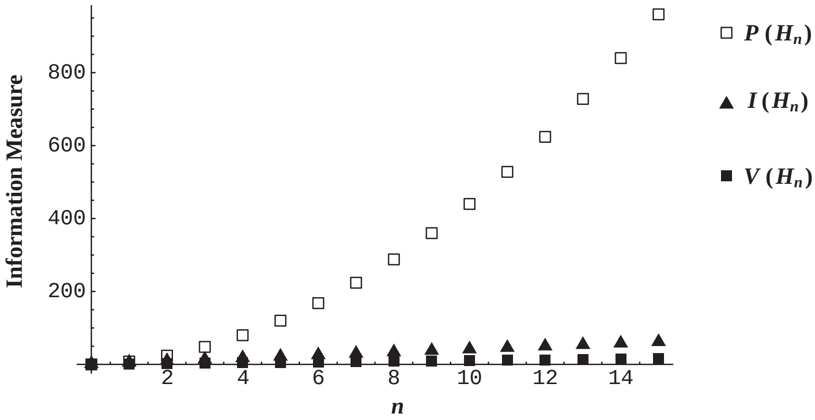

Fig. 2. Information measures of the Hermite polynomials $H_n(x)$. The variation of the Fisher information (triangles), the variance (filled boxes), and the information product (blank boxes) is analyzed.

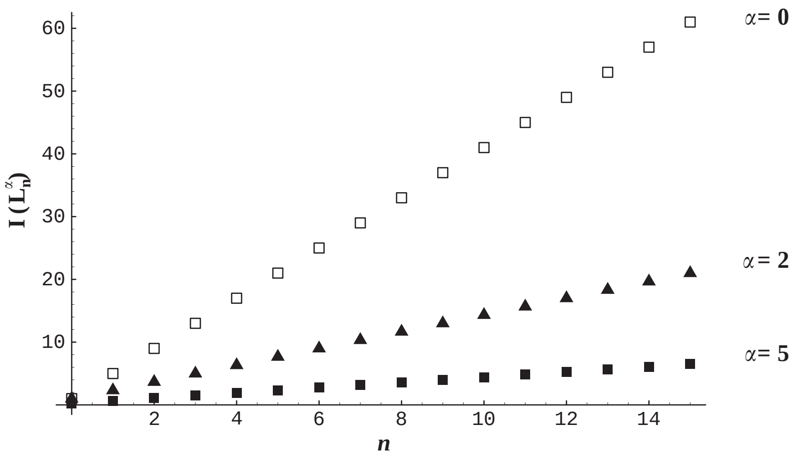

Fig. 3. Variation with the degree n of the Fisher information of the Laguerre polynomials $L_n^{\alpha}(x)$, $\alpha = 0$, 2 and 5.

indicating a similar oscillatory behavior (as one can see by the comparison of the Rakhmanov densities of some Hermite and Laguerre polynomials shown by Figs. 1 and 5, respectively). The variance, see Fig. 4, depends quadratically on the degree n, indicating that the Rakhmanov probability spreads around the centroid when n is increasing much faster than in the Hermite case. For completeness let us point out that in contrast to the Hermite case, we have that (a) the centroid is no more the origin for all values of n, but it shifts towards higher values when n is increasing, and (b) the probability mass is not distributed symmetrically around the centroid, and it behaves so that not only the number of peaks increases but also, the peaks near the origin (which is an extreme of orthogonality interval) get much sharper when the degree n is increasing.

The information product $\mathscr{P}(L_n^{\alpha})$ for $\alpha = 0$, 2 and 5 is analyzed in Fig. 6, where we observe its cubic dependence with respect to the degree due to the multiplicative effect of the variance on the Fisher

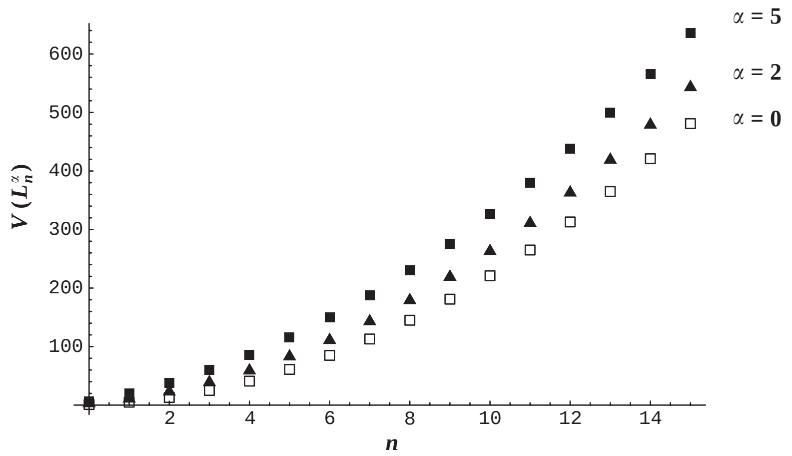

Fig. 4. Dependence of the variance on the degree n for the Laguerre polynomials $L_n^\alpha(x)$, $\alpha = 0, 2$ and 5.

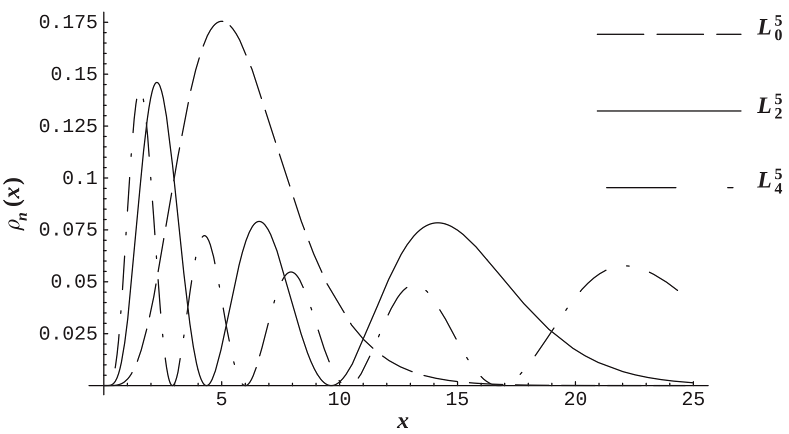

Fig. 5. Rakhmanov densities of the Laguerre polynomials $L_n^\alpha(x)$, with $\alpha = 5$ and $n = 0, 2$ and 4.

information. For completeness let us mention that for $\alpha = 0$ we have that

$$I(L_n) = 4n + 1, \quad V(L_n) = 2n^2 + 2n + 1,$$

so that the corresponding region in the Cramer–Rao information plane is a discrete root-squared line.

It is also interesting to study the variation of the information measures of the Laguerre polynomials of a given degree n, $L_n^\alpha(x)$, with respect to the parameter α. These quantities allow to quantitatively estimate the dependence of the oscillatory frequency and the spreading of the Rakhmanov density of the Laguerre polynomials $\rho_n(x;\alpha) = [L_n^\alpha(x)]^2 x^\alpha \exp(-x)$ (and of the polynomials themselves) when the parameter α is varying; see various densities of this type for the polynomials $L_0^\alpha(x)$ in Fig. 7, and for $L_3^\alpha(x)$ in Fig. 8, when α=1, 3 and 5. The results obtained for the Fisher information of various Laguerre polynomials are shown, according to Eq. (10), in Fig. 9. Therein we observe that the Fisher information of the

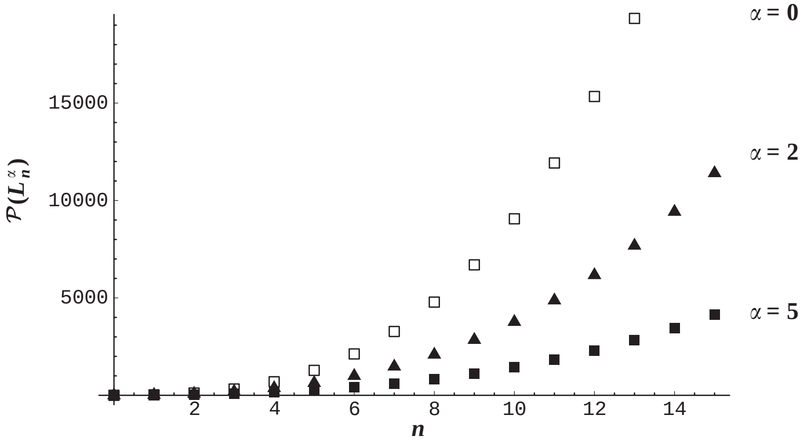

Fig. 6. Dependence on the degree n of the information product $\mathscr{P}(L_n^\alpha)$ of the Laguerre polynomials with $\alpha = 0$, 2 and 5.

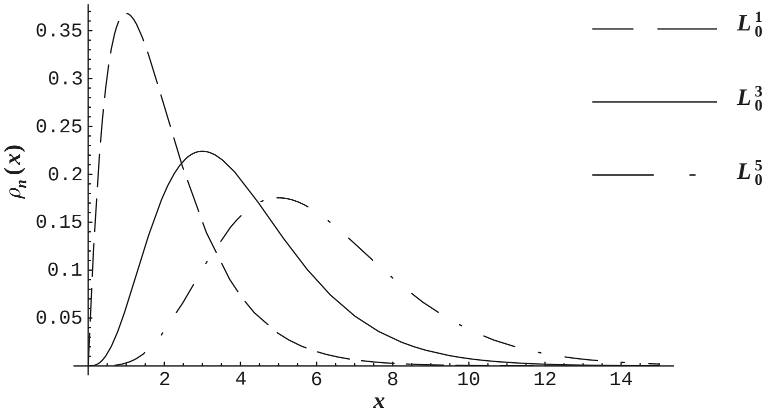

Fig. 7. Rakhmanov density of the Laguerre polynomials $L_0^\alpha(x)$ with $\alpha = 1$, 3 and 5.

Laguerre polynomials with a given degree decreases to zero for $\alpha > 1$ in a very fast and monotonic way. It behaves as

$$I(L_n^\alpha) = \frac{2n}{\alpha} + O(1), \quad \alpha > 1,$$

for large values of α. This behavior can be explained by looking at Figs. 7 and 8. We see that there is an enhancement effect of smoothness for the probability mass (and for the polynomials themselves), so that the peaks get wider and smoother when the parameter α is increasing, what makes the derivative smaller, provoking the decrease of the Fisher information. Moreover, in these two figures we notice that the origin plays a repulsion role on the probability mass in such a way that the variance of the polynomials linearly increases when α is increasing as described by Eq. (16).

What about the information product $\mathscr{P}(L_n^\alpha)$? How does it depend on the parameter α? This is analyzed in Fig. 10. There in we observe that it rapidly decreases when α is increasing so that it tends towards a

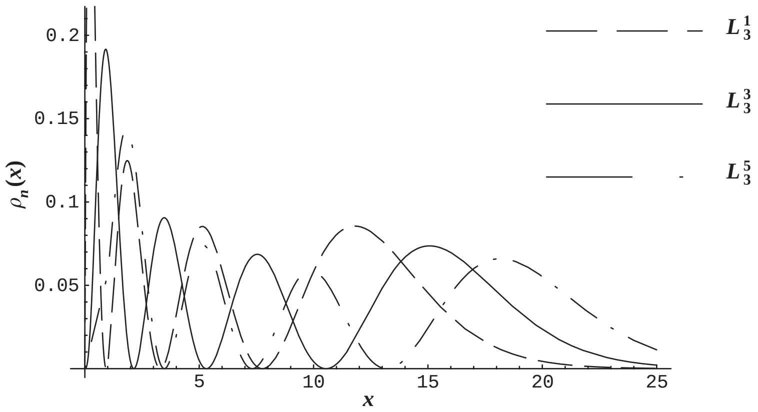

Fig. 8. Rakhmanov density of the Laguerre polynomials $L_3^\alpha(x)$ with $\alpha = 1, 3$ and 5.

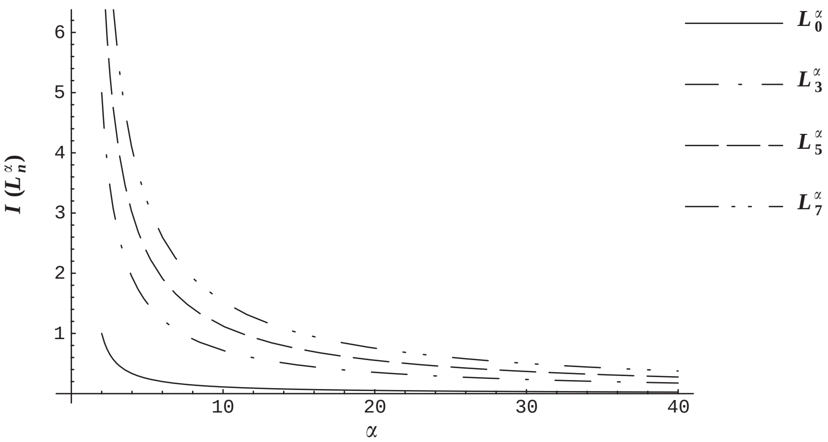

Fig. 9. Variation with the parameter α ($\alpha > 1$) of the Fisher information of the Laguerre polynomials $L_n^\alpha(x)$ with $n = 0, 3, 5$ and 7.

constant limit other than zero; namely

$$\mathscr{P}(L_n^\alpha) \sim 2n(2n+1), \quad \alpha \gg 1.$$

The Cramer–Rao information plane of the Laguerre polynomials given by Fig. 11, gathers the variation of the previous information measures with respect to the degree n and the parameter α. Only the polynomials with $\alpha = 0$ and $\alpha > 1$ are considered in the plane, since the Fisher information is not defined for the rest. First, we observe that we can determine the polynomial with a given couple of values (I, V); in fact we can find from Eqs. (10) and (16) the degree n and the parameter α in a straightforward but a bit cumbersome manner. Second, the information-theoretic behavior of the Laguerre polynomial with $\alpha = 0$, shown by a discrete and concave line, clearly differs from that of the rest of polynomials with $\alpha > 1$. Indeed, the (I, V) values of the polynomial $L_n^0(x)$ lie to the left of the convex continuous line of values of the polynomials $L_n^\alpha(x)$ with $\alpha > 1$, indicating that the former polynomial has a smaller variance so that

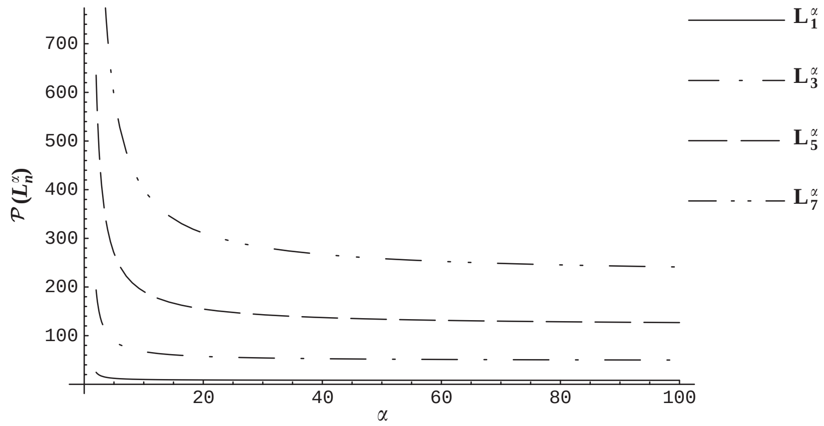

Fig. 10. Information product $\mathscr{P}(L_n^\alpha)$ of the Laguerre polynomials as a function of the degree n and the parameter α. It tends to the limit $2n(2n+1)$ when $\alpha \to \infty$.

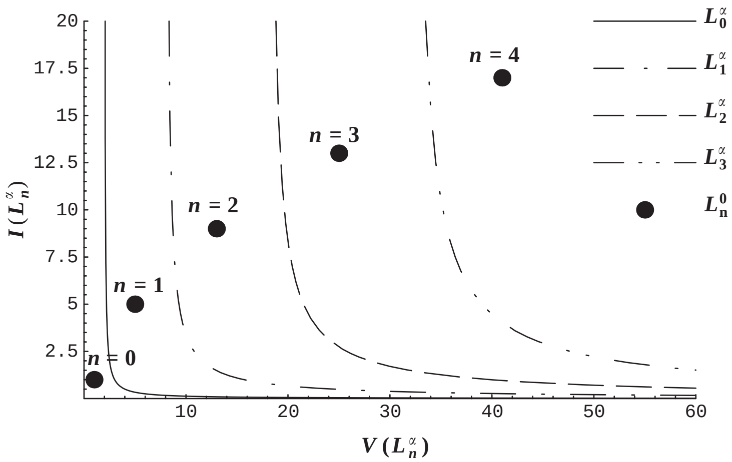

Fig. 11. Cramer–Rao information plane of the Laguerre polynomials L_n^0 with $n = 0 - 4$, and $L_n^\alpha(x)$ with $n = 0 - 3$ and $\alpha > 1$.

it is spread much less around the centroid than the latter polynomials. Third, in the general case $\alpha > 1$ we observe that (i) the Fisher information of $L_n^\alpha(x)$ with n fixed is a continuous monotonically decreasing function of the variance, (ii) the continuous lines of the plane move to the right when n is increasing.

4.3. *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x),\ \alpha > -1,\ \beta > -1,\ x \in [-1, +1]$

The Fisher information of the Jacobi polynomials is finite only when (a) $\alpha = \beta = 0$, (b) $\alpha = 0$ and $\beta > 1$, and, symmetrically, $\alpha > 1$ and $\beta = 0$, and (c) $\alpha > 1$ and $\beta > 1$, as follows from Eq. (11). In all these cases it is found a cubic dependence with the degree n; see Fig. 12, where such a variation is explicitly shown for some specific values of the pair (α, β). This behavior indicates that the corresponding Rakmanov density

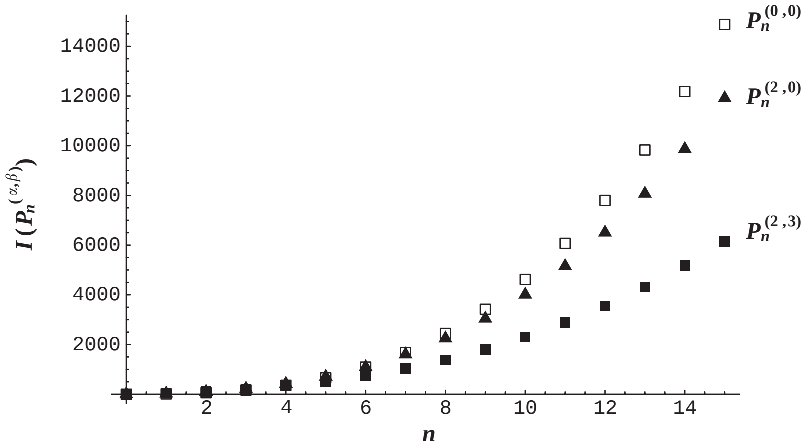

Fig. 12. Dependence of the Fisher information of the Jacobi polynomials $\mathscr{P}_n^{(\alpha,\beta)}(x)$ on the degree n for the pairs of parameters $(\alpha, \beta) = (0, 0), (2, 0), (2, 3)$.

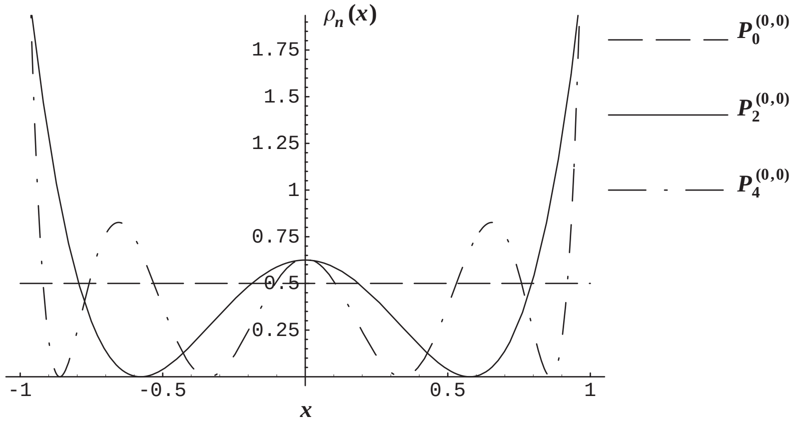

Fig. 13. Rakhmanov density of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \beta = 0$ and $n = 0, 2$ and 4.

shows, when the degree n is increasing, bias to more and more particular x values, (i.e., it is steeply sloped about these x values), indicating that the probability mass is more concentrated, so less disordered. Figs. 13–15 illustrates this phenomenon in the Rakhmanov density of specific polynomials of the previous types (a), (b) and (c). For completeness let us mention that for large values of n the Fisher information $I(P_n^{(\alpha,\beta)})$ behaves, according to Eq. (11), as

$$I(P_n^{(\alpha,\beta)}) = \begin{cases} \dfrac{(\alpha\beta - 1)(\alpha + \beta)}{(\alpha^2 - 1)(\beta^2 - 1)} n^3 + O(n^2) & \text{for } \alpha, \beta > 1, \\ \dfrac{\alpha - 2}{\alpha - 1} n^3 + O(n^2) & \text{for } \alpha = 0, \beta > 1, \\ 4n^3 + O(n^2) & \text{for } \alpha, \beta = 0. \end{cases} \tag{18}$$

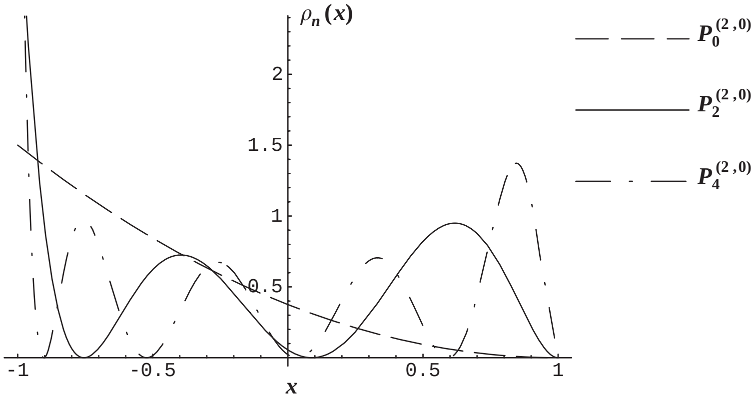

Fig. 14. Rakhmanov density of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = 2$, $\beta = 0$ and $n = 0$, 2 and 4.

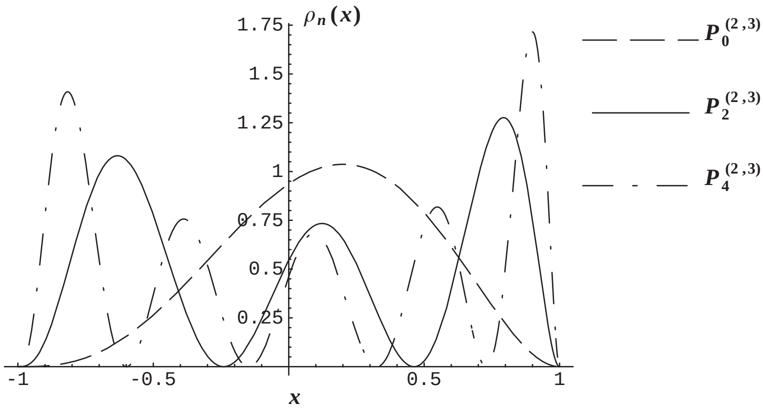

Fig. 15. Rakhmanov density of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = 2$, $\beta = 3$ and $n = 0$, 2 and 4.

The variance of the polynomials, $V(P_n^{(\alpha,\beta)})$, is always defined as one can see in Eq. (17). Its dependence on the degree n for some given pairs of parameters (α, β) is plotted in Fig. 16. We observe that this quantity (i) slowly decreases down to 1/2 when $\alpha = \beta = 0$, save for the special case $n = 0$ where its value is 1/3 (uniform distribution), and (ii) slowly increases up to the asymptotical value 1/2 for the remaining pairs (α, β), so that for n fixed, the maximum value of the variance is obtained in the Gegenbauer case with $\alpha = \beta = 1$.

The variation of the information product $\mathscr{P}(P_n^{(\alpha,\beta)})$ with the degree n is analyzed for various pairs of parameters (α, β) in Fig. 17. Therein we observe that this quantity is an increasing function of the degree n, which behaves as n^3 for large values of n.

Let us now study the behavior of the three previous information measures with the parameters (α, β). The analysis of the variation of the Fisher information $I(P_n^{(\alpha,\beta)})$ with the parameter α for (n, β) fixed is done, following Eq. (11), in Fig. 18 where $\beta = 0$ and 3 and $n = 0$, 3, 5 and 7 were taken. We observe that the Fisher information has a parabolic behavior with α $(\alpha > 1)$, having the equilibrium point at $\alpha = \alpha_e$,

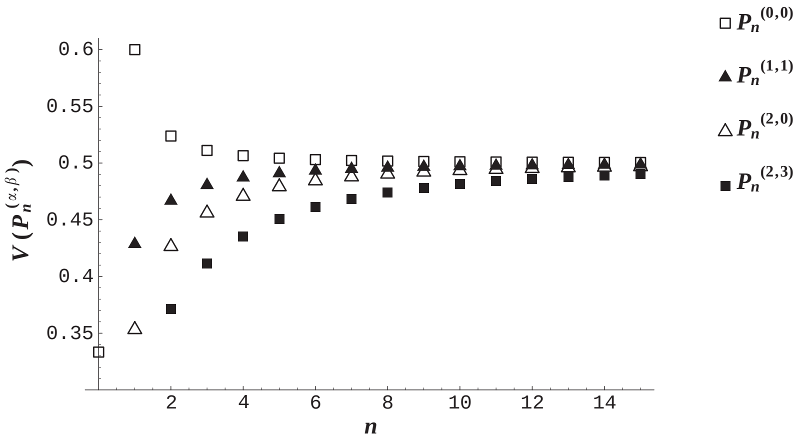

Fig. 16. Dependence of the variance of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ on the degree n for the parameters $(\alpha, \beta) = (0, 0), (2, 0), (1, 1)$ and $(2, 3)$.

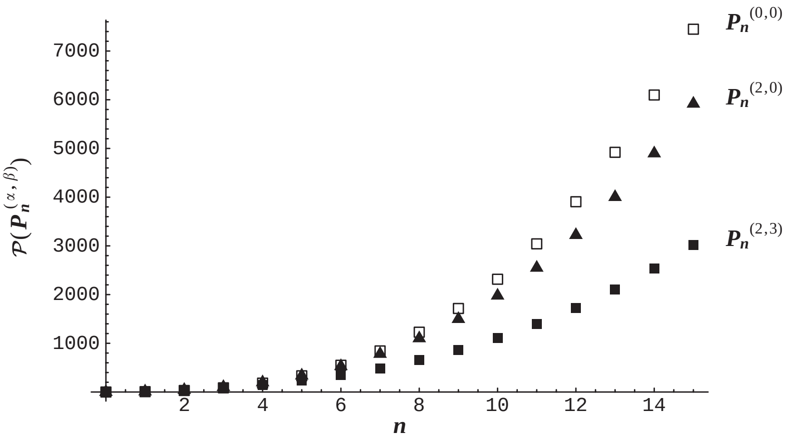

Fig. 17. Information product of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for the parameters $(\alpha, \beta) = (0, 0), (2, 0)$ and $(2, 3)$.

whose precise value can be numerically obtained as the root of a polynomial which can be found for given β and n in an analytical form. Remark that this critical value shifts to larger values when n is increasing. Then, one has that for $\alpha \in (1, \alpha_e]$ the Fisher information rapidly decreases, indicating that the Rakmanov density $\rho_n(x; \alpha, \beta)$, so as to the polynomials themselves, gets broader and smoother; and for $\alpha \in (\alpha_e, \infty)$ the Fisher information continuously increases, what indicates that both the corresponding polynomials and the probability mass show a stronger oscillatory character and less disorder.

The behavior of the variance $V(P_n^{(\alpha,\beta)})$ with the parameter α ($\alpha > 1$) for (n, β) fixed is done, according to Eq. (17), in Fig. 19 where $\beta = 0$ and 3 and $n = 0, 3, 5$ and 7 were taken for the sake of illustration. We observe that this quantity is a concave function of α with the maximum at $\alpha = \alpha_{\max}$, whose value can be obtained similarly as the previous α_e. Notice that this maximum (i) shifts towards lower α values when n is increasing and β is fixed, and (ii) goes to higher α values when n is fixed and β is increasing.

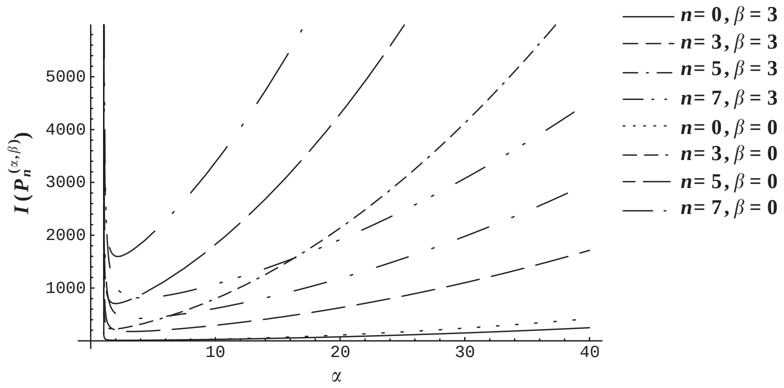

Fig. 18. Variation with the parameter α ($\alpha > 1$) of the Fisher information of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for (n, β) fixed. The functions $I(P_n^{(\alpha,0)})$ and $I(P_n^{(\alpha,3)})$ were plotted for $n = 0$, 3, 5 and 7.

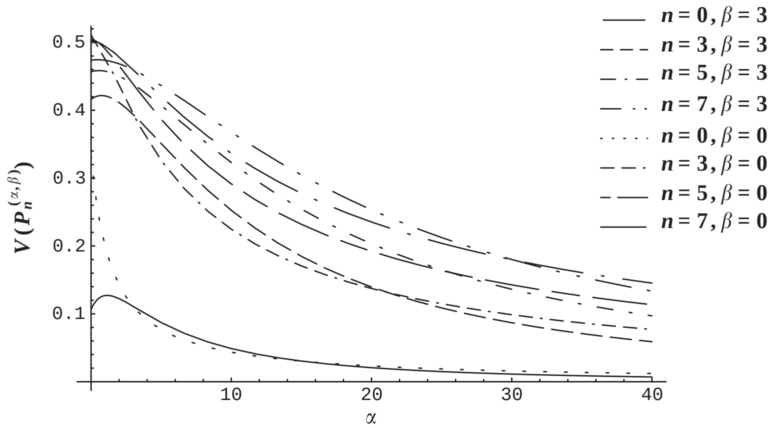

Fig. 19. Variation with the parameter α ($\alpha > 1$) of the variance of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ for (n, β) fixed. The functions $V(P_n^{(\alpha,0)})$ and $V(P_n^{(\alpha,3)})$ were plotted for $n = 0$, 3, 5 and 7.

The information product $\mathscr{P}(P_n^{(\alpha,\beta)})$ behaves with the parameter α ($\alpha > 1$) for (n, β) fixed as shown in Fig. 20. It is observed that, in contrast to the increasing (decreasing) behavior of the Fisher information (variance) with respect to α, the information product $\mathscr{P}$ allows us to disentangle the two cases $\beta = 0$ and $\beta > 1$. Indeed, the dependence of the information product on the parameter α is linearly increasing for $\beta = 0$, but it tends to a constant (which can be determined analytically) for $\beta > 1$, then showing a nearly horizontal behavior like that we have already seen in the Laguerre case $L_n^{\alpha}(x)$ for $\alpha > 1$.

The mutual relationship between the Fisher information and the variance of the Jacobi polynomials can be discussed in detail by means of the Cramer–Rao information plane. This plane is plotted in Fig. 21 for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in the following cases: (i) $n = 0 - 4$ and $\alpha = \beta = 0$, (ii) $n = 0$, 1, 2 and $\alpha > 1$, $\beta = 0$, and (iii) $n = 0$, 1 and 2, and $\alpha > 1$, $\beta > 1$. We observe a discrete line of points, three continuous lines, and three partially overlapping full regions in the three cases, respectively. It is

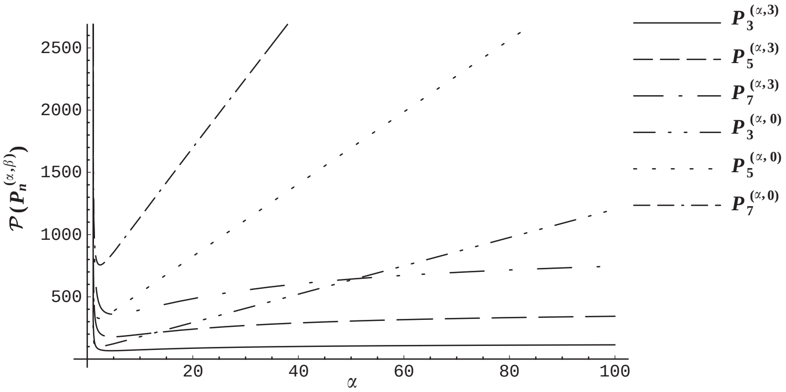

Fig. 20. Variation with the parameter α ($\alpha > 1$) of the information product of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for (n, β) fixed. The functions $\mathscr{P}_n^{(\alpha,0)}(x)$ and $\mathscr{P}_n^{(\alpha,3)}(x)$ were plotted against α for $n = 3, 5$ and 7.

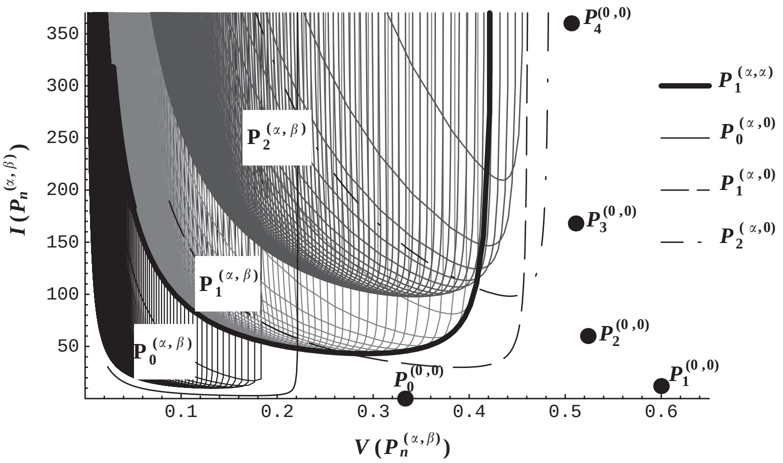

Fig. 21. Cramer–Rao information plane of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for $n = 0$, 1 and 2. The discrete lines for $\alpha = \beta = 0$, the continuous lines with $\alpha > 1$ and $\beta = 0$, and the three regions with $\alpha > 1$ and $\beta = 3$ are plotted.

interesting to remark that the line which envelopes the full regions corresponds to the (I, V) points of the corresponding Gegenbauer case $\alpha = \beta$.

5. Conclusions

Beyond its mathematical [15] and computational [4] interest, the Rakhmanov density of the classical orthogonal polynomials describes the quantum-mechanical probability density of the physical states of numerous one-body and many-body systems in an exact or approximate manner, respectively [7,13,14]. In this paper, the spreading of the Rakhmanov density of the classical orthogonal polynomials on a real interval (i.e., Hermite, Laguerre and Jacobi) all over the orthogonality domain is discussed both

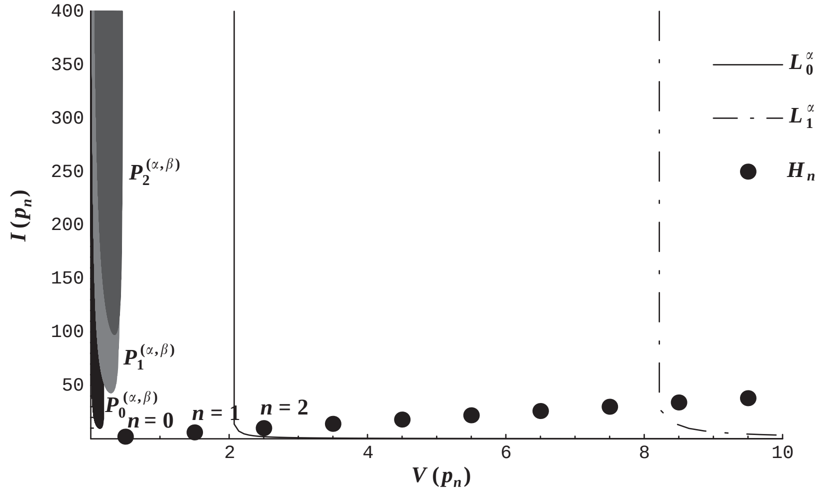

Fig. 22. Cramer–Rao information plane of the classical orthogonal polynomials $p_n(x)$: the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for various values of the degree n and the parameters. The discrete line corresponds to the (I, V) values of the Hermite polynomials $H_n(x)$ with degrees $n = 0 - 9$. The two continuous lines refers to the two Laguerre polynomials $L_n^{\alpha}(x)$ with lowest degrees $n = 0$ and 1, and with parameters $\alpha > 1$. The three full regions just near the origin corresponding from the left to the right to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with degrees $n = 0$, 1 and 2, and parameters $\alpha > 1$, $\beta > 1$.

analytically and numerically by means of three information-theoretic quantities of local (Fisher information I), global (variance V) and mixed (information product $\mathscr{P}$) character. The Fisher information measures the gradient content of the density, providing a quantitative estimation of the oscillatory character of the density and the polynomials, and the bias to particular x values in the interval; i.e., it measures the degree of local disorder. The variance gives a quantitative measure of the broadness of the density around the centroid. These two information-theoretic measures are somewhat the two faces of the same coin: one refers to concentration of the probability mass on specific x-values of the orthogonality interval, and the other supplies the central spreading of it. In an attempt to grasp both characters at the same time, we have introduced and discussed a new measure, the Fisher-variance information product, which has two advantages: the nonnegativeness (as the Fisher information and the variance) and the scale invariance (as the Kullback–Leibler or relative entropy).

Our results can be most conveniently gathered in the so-called Cramer–Rao information plane of the classical orthogonal polynomials. For illustration this plane is shown in Fig. 22, which includes the (Fisher information, variance) values of the Hermite polynomials $H_n(x)$ with $n = 0 - 9$ (shown by the discrete line of bullets), the Laguerre polynomials $L_n^{\alpha}(x)$ with $n = 0$, 1 and $\alpha > 1$ (see the two continuous lines), and the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $n = 0 - 2$ and $\alpha > 1$, $\beta > 1$ (as shown by the three partially overlapping full regions in the neighborhood of the origin).

Finally let us mention some open related problems in the emerging information theory of special functions, which underlies the modern information theory of solvable and quasi-solvable quantum-mechanical systems. First, to pose new characterization problems for the classical orthogonal polynomials by means of their information-theoretic measures. Second, to extend the covering of the Cramer–Rao information plane by the remaining orthogonal polynomials of the Askey tableau. Third, to find the Cramer–Rao information plane of the classical orthogonal polynomials in a discrete variable. Fourth, to define and discuss the Fisher–Shannon information plane for all the previous orthogonal polynomials. We believe that the solution of these problems can contribute to the development of the information theory of orthogonal polynomials, but also to the comprehension of the internal disorder of numerous quantum systems.

Acknowledgements

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